in which p stands for dy/dx. *Apparently* (f) is *independent* of the law of resistance of the medium in the direction opposite to motion.

Assuming that the co-ordinate axis OX makes an angle ω with a horizontal plane, we have from the parallelogram of *velocity-components*

$$\cos(90-\omega)=\frac{(dy/dt)^2+(dx/dt)^2-(ds/dt)^2}{2(dy/dt)(dx/dt)}\ldots\ldots(g).$$

$$\therefore \left(\frac{ds}{dt}\right)^2=\left(\frac{dy}{dt}\right)^2-2\left(\frac{dy}{dt}\right)\left(\frac{dx}{dt}\right)\sin\omega+\left(\frac{dx}{dt}\right)^2$$

$$=\left(\frac{dx}{dt}\right)^2\left[\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)\sin\omega+1\right]=\left(\frac{dx}{dt}\right)^2[p^2-2p\sin\omega+1]\ldots\ldots(h).$$

From (a) in (3), by means of (h), may be deduced

$$\frac{d^2x}{dt^2}=-\frac{g}{w^3}\left(\frac{ds}{dt}\right)^2\left(\frac{ds}{dt}\times\frac{dx}{ds}\right)=-\frac{g}{w^3}\left(\frac{ds}{dt}\right)^2\frac{dx}{dt}$$

$$=-\frac{g}{w^3}\left(\frac{dx}{dt}\right)[p^2-2p\sin\omega+1]\ldots\ldots(i).$$

Eliminating g from (i), by means of (f),

$$\frac{dx}{dt}=-\frac{1}{w^3}\left(\frac{dp}{dt}\cdot\frac{dx}{dt}\right)\left(\frac{dx}{dt}\right)^3\left[\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)\sin\omega+1\right]\ldots.(j).$$

$$\therefore w^3\left(\frac{dx}{dt}\right)^{-4}\frac{d^2x}{dt^2}=-\left[\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)\sin\omega+1\right]\frac{d}{dt}\left(\frac{dy}{dx}\right)\ldots.(k).$$

Remembering that when $dy/dx=0$, $dx/dt=x$, and integrating (k),

$$\tfrac{1}{3}w^3(dx/dt)^{-3}=\tfrac{1}{3}p^3-p^2\sin\omega+p\ldots.(l).$$

$$\therefore dx/dt=w/\sqrt[3]{(p^3-3p^2\sin\omega+3p)}\ldots.(m).$$

Obvious transformations of (f) give

$$\frac{dp}{dt}=g\left(\frac{dx}{dt}\right)^{-1},\text{ or }\frac{dp}{dx}\times\frac{dt}{dx}=g\left(\frac{dx}{dt}\right)^{-2}\ldots.(a').$$

$$\therefore dp/dx=(g/w^3)[p^3-3p^2\sin\omega+3p]^{\frac{1}{3}}\ldots.(n).$$

Taking reciprocals of (n), etc.,

$$(g/w^2)(dx/dp)=1/[\,p^3-3p^2\sin\omega+3p]^{\frac{1}{2}}\ldots.(o).$$

Multiplying the left-hand member of (o) by dy/dx and the right-hand member by p, etc.,

$$(g/w^2)(dy/dp)=p/[\,p^3-3p^2\sin\omega+3p]^{\frac{1}{2}}\ldots.(p).$$

From (f) and (m), can easily be deduced

$$\frac{dp}{dt}=g/\frac{dx}{dt}=(g/w)[\,p^3-3p^2\sin\omega+3p]^{\frac{1}{2}}\ldots.(q).$$

Also, from (o), (p), and (q), respectively, by obvious integrations, etc.,

$$\frac{gx}{w^2}=\int_0^\infty d\left(\frac{dy}{dx}\right)/\left[\left(\frac{dy}{dx}\right)^3-3\left(\frac{dy}{dx}\right)^2\sin\omega+3\left(\frac{dy}{dx}\right)\right]^{\frac{1}{2}}\ldots.(r),$$

$$\frac{gy}{w^2}=\int_0^\infty\frac{dy}{dx}d\left(\frac{dy}{dx}\right)/\left(\frac{dy}{dx}\right)^3-3\left(\frac{dy}{dx}\right)^2\sin\omega+3\left(\frac{dy}{dx}\right)\Big]^{\frac{1}{2}}\ldots.(s),$$

$$\frac{gt}{w}=\int_0^\infty d\left(\frac{dy}{dx}\right)/\left[\left(\frac{dy}{dx}\right)^3-3\left(\frac{dy}{dx}\right)^2\sin\omega+3\left(\frac{dy}{dx}\right)\right]^{\frac{1}{2}}\ldots.(t).$$

Assume $z=M^2\sqrt[3]{(\,p^3-3p^2\sin\omega+3p)/p}\ldots.(\zeta')$.

Cubing (ζ'), differentiating, etc.,

$$6z^2dz=6M^6[(\,p\sin\omega-2)/p^3]dp\ldots.(\gamma').$$

From (ζ'), after cubing, can be formed the expression,

$$4z^3-g_3=[(4M^6-g_3)p^2-12M^6p\sin\omega+12M^6]/p^2\ldots.(\delta'),$$

which is a perfect square when we have $4M^6-g_3=3M^6\sin^2\omega$, or when $g_3=M^6(4-3\sin^2\omega)$. Substituting this value of g_3 in (δ), etc.,

$$\sqrt{(4z^3-g_3}=M^3(\,p\sin\omega-2)\sqrt{(3)}/p\ldots.(\epsilon').$$

Dividing (γ') by (ε'), etc., also making the *arbitrary* constant factor $M^2=\frac{1}{3}$, we have

$$\frac{6z^2dz}{\sqrt{(4z^3-g_3)}}=\frac{6M^6(\,p\sin\omega-2)dp}{p^3}\times\frac{p}{M^3(\,p\sin\omega-2)\sqrt{3}}\ldots.(\zeta').$$

$$\therefore \frac{dz}{\sqrt{(4z^3-g_3)}}=\frac{M^3\sqrt{(3)}}{3p^2z^2}\times\frac{g[\,p^3-3p^2\sin\omega+3p\,]^{\frac{1}{2}}\,dx}{w^2}$$

$$=\frac{M^3\sqrt{(3)}\times p^2}{3p^2\times M^4[\,p^3-3p^2\sin\omega+3p\,]^{\frac{1}{2}}}\times\frac{g[\,p^3-3p^2\sin\omega+3p\,]^{\frac{1}{2}}\,dx}{w^2}\ldots.(\eta').$$

$$\therefore \frac{gx}{w^2}=\int_0^{\infty}\frac{dz}{\sqrt{(4z^3-g_3)}},=*(p,u)^{-1}(z;0,g_3)\ldots.(\theta').$$

That is, $z=*(p,u)\left(\frac{gx}{w^2};0,g_3\right)\ldots.(\iota')$,

which is an *incomplete* form of the *first* of the three Weierstrassian functions by means of which it is possible to express any elliptic integral; and this function is expressed in its *inverted* form according to the Abelian spstem of notation. Now, $*(p,u)'\left(\frac{gx}{w^2}\right)=\frac{p\sin\omega-2}{3p}$; and if $z=a$ at the vertical asymptote, where $p=\infty$, we must have

$$*(p,u)'\left(\frac{ga}{w^2}\right)=\frac{\sin\omega}{3},\text{ and }*(p,u)\left(\frac{ga}{w^2}\right)=\tfrac{1}{3}.$$

$$\therefore *(p,u)'\left(\frac{ga}{w^2}\right)-*(p,u)'\left(\frac{gx}{w^2}\right)=\frac{2}{3p}\ldots.(\kappa').$$

From (κ') by obvious transformations

$$p=\frac{dy}{dx}=\tfrac{2}{3}/\left[*(p,u)'\left(\frac{ga}{w^2}\right)-*(p,u)'\left(\frac{gx}{w^2}\right)\right]\ldots.(\lambda').$$

$$\therefore y=\int_0\left[6*(p,u)^2\left(\frac{ga}{w^2}\right)/\left[*(p,u)'\left(\frac{ga}{w^2}\right)-*(p\text{-}u)'\left(\frac{gx}{w^2}\right)\right]\right]dx\ldots.(\mu'),$$

which is the equation of the trajectory expressed according to the Weierstrassian system of functional notation.

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*Being without the proper character for the (p,u) function we designate it by (p,u).

APPLICATIONS OF A THEOREM REGARDING CIRCULANTS.

By HARRY S. VANDIVER, Bala, Pa.

A circulant of the nth order is a simple determinant of the following type:

$$\begin{vmatrix} a_1 & \omega_n a_2 & \omega_n^2 a_3 & \cdots & \omega_n^{n-1} a_n \\ \omega_n^{n-1} a_n & a_1 & \omega_n a_2 & \cdots & \omega_n^{n-2} a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega_n a_2 & \omega_n^2 a_3 & \omega_n^3 a_4 & \cdots & a_1 \end{vmatrix}$$

where ω_n is a special nth root of unity. This is the product of n linear factors, as follows:

$$\begin{aligned} &a_1 + \omega_n a_2 + \omega_n^2 a_3 + \omega_n^3 a_4 \cdots \omega_n^{n-1} a_n \\ &a_1 + \omega_n^2 a_2 + \omega_n^4 a_3 + \omega_n^6 a_4 \cdots \omega_n^{n-2} a_n \\ &a_1 + \omega_n^3 a_2 + \omega_n^6 a_3 + \omega_n^9 a_4 \cdots \omega_n^{n-3} a_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_1 + a_2 + a_3 \cdots\cdots\cdots\cdots a_n \end{aligned}$$

A theorem regarding such functions is:

The product of any number of circulants each of the order n can be expressed as a circulant of the nth order.

For simplicity, we will prove this for continuants of the third order; the procedure is precisely the same when $n>3$. We have:

$$\begin{vmatrix} a_1 & \omega_3 a_2 & \omega_3^2 a_3 \\ \omega_3^2 a_3 & a_1 & \omega_3 a_2 \\ \omega_3 b_2 & \omega_3^2 a_3 & a_1 \end{vmatrix} \times \begin{vmatrix} b_1 & \omega_3 b_2 & \omega_3^2 b_3 \\ \omega_3^2 b_3 & b_1 & \omega_3 b_2 \\ \omega_3 b_2 & \omega_3^2 b_3 & b_1 \end{vmatrix}$$

This equals the product of the six factors:

$$\begin{aligned} &(a_1 + a_2 + a_3)(b_1 + b_2 + b_3) \\ &(a_1 + \omega_3 a_2 + \omega_3^2 a_2)(b_1 + \omega_3 b_2 + \omega_3^2 b_3) \\ &(a_1 + \omega_3^2 a_2 + \omega_3 a_3)(b_1 + \omega_3^2 b_2 + \omega_3 b_3) \end{aligned}$$

Taken two and two, these give:

$$\begin{aligned} &A_1 + A_2 + A_3 \\ &A_1 + \omega_3 A_2 + \omega_3^2 A_3 \\ &A_1 + \omega_3^2 A_2 + \omega_3 A_3. \end{aligned}$$ Whence the theorem.

If it is true for two circulants it is true for any number.

Corollary. Any integral power of a circulant of the nth order is a circulant of the same order.

Some of the applications of this theorem and corollary to Diophantine analysis follow:

$$\begin{vmatrix} x & iy \\ iy & x \end{vmatrix} \times \begin{vmatrix} a & ib \\ ib & a \end{vmatrix} = \begin{vmatrix} c & id \\ id & c \end{vmatrix}$$

or $(x^2+y^2)(a^2+b^2)=c^2+d^2$, a well known theorem.

If the relation $x^2+bxy+cy^2=1$ possesses one solution in integers it possesses an infinite number.

The function $x^2+bxy+cy^2$ can be expressed as a circulant as follows:

$$\begin{vmatrix} \frac{2x+by}{2} & \frac{1}{2}\sqrt{(b^2-ac)}\,y \\ \frac{1}{2}\sqrt{(b^2-4ac)}\,y & \frac{2x+by}{2} \end{vmatrix}$$

Suppose the equation possesses one solution, say $x=m$, $y=n$.

Then $m^2+bmn+cn^2=1$; then we can put

$$(m^2+bmn+cn^2)^k=x^2+bxy+cy^2.$$

Putting $k=2$, and decomposing both circulants into their irrational factors, there is obtained:

$$\frac{2x+by-y\sqrt{(b^2-4c)}}{2}=\left[\frac{2m+bn-n\sqrt{(b^2-4c)}}{2}\right]^2$$

and

$$\frac{2x+by+y\sqrt{(b^2-4c)}}{2}=\left[\frac{2m+bn+n\sqrt{(b^2-4c)}}{2}\right]^2$$

Taking the first relation and expanding

$$4x+2by-2y\sqrt{(b^2-4c)}=(2m+bn)^2-2n(2m-bn)\sqrt{(b^2-4c)}+n^2(b^2-4c).$$

Equating rational and irrational parts

$$4x+2by=(2m+bn)^2+n^2(b^2-4c) \text{ and } y=n(2m+bn).$$

By substitution we find $x=m^2-cn^2$.

Proceeding in the same manner when $k>2$, we find integral values for x and y. Hence since k is unlimited the number of solutions is unlimited. Hence the theorem.

To show that, for every value of n, the relation

$$x^3+ay^3+a^2z^3-3axyz=v^n$$

has (when a is an integer) an infinite number of integral solutions in x, y, and z.

The function $x^3+ay^3+a^2z^3-3axyz$ expressed as a circulant is:

$$\begin{vmatrix} x & y\sqrt[3]{a} & z\sqrt[3]{a^2} \\ z\sqrt[3]{a^2} & x & y\sqrt[3]{a} \\ y\sqrt[3]{a} & z\sqrt[3]{a^2} & x \end{vmatrix}$$

We then can assume $(x_1^3+ay_1^3+a^2z_1^3-3ax_1y_1z_1)^n=x^3+ay^3+a^2z^3-3axyz$.

Put $n=2$. Then

$$(x_1+y_1\sqrt[3]{a}+z_1\sqrt[3]{a^2})^2=x+y\sqrt[3]{a}+z\sqrt[3]{a^3}.$$

From which, by expanding and equating irrationals,

$$x=x_1^2+2ay_1z_1$$
$$y=z_1^2a+2x_1y_1$$
$$z=y_1^2+2x_1y_1.$$

The number of values that can be assigned to x_1, y_1, z_1 is infinite; hence the theorem is proved for $n=2$. When $n>2$ it may be proved by proceeding in the same manner.

In Article 9, entitled, "Of the manner of finding Algebraic Functions of all degrees, which when multiplied together may always produce similar functions" in the Additions to Euler's Algebra, Lagrange gives developments which may be considered as applications of the circulant theorem.

Bala, Pa., July 13, 1901.

THE BETWEENNESS ASSUMPTIONS.

By DR. GEORGE BRUCE HALSTED.

In his "Vorlesung ueber Euklidische Geometrie" (Wintersemester 1898/99), Hilbert gave the most remarkable set of axioms ever created for the founding of geometry. These were published in his Festschrift, 1899, treated of in my "Supplementary Report" to the A. A. A. S., (*Science*, Nov. 8, 1901). They have been given in English, though in part erroneously, in D. E. Smith's "The Teaching of Elementary Mathematics," (Macmillan).

The first group of these assumptions Hilbert calls *Axioms of Association*. Of these we need only mention—

I 1. Two distinct points, A, B, always determine a straight a.

I 7. On every straight there are at least two points; in every plane there are at least three non-co-straight points; and in space there are at least four non-co-straight, non-co-planar points.

The second group of assumptions Hilbert calls *Axioms of Arrangement*.

It is these that I call the *Betweenness Assumptions*.

Of them Hilbert says in §3: "The axioms of this group define the idea "between," and make possible on the basis of this idea the *arrangement* of the points on a straight, in a plane and in space.

Convention. The points of a straight stand in certain relations to one another, to describe which especially the word "*between*" serves us.

II 1. If A, B, C are points of a straight, and B lies between A and C, then B also lies between C and A.

II 2. If A and C are two points of a straight, then there is always at least one point B, which lies between A and C, and at least one point, such that C lies between A and D.

II 3. Of any three points of a straight there is always one and only one, which lies between the other two.

II 4. Any four points A, B, C, D of a straight can always be so arranged that B lies between A and C and also between A and D, and furthermore C lies between A and D and also between B and D.

DEFINITION. The system of two points A and B, which lie upon a straight a, we call a *sect*, and designate it with AB or BA.

The points between A and B are said to be points of the sect AB or also situated *within* the sect AB; all remaining points of the straight a are said to be situated *without* the sect AB. The points A, B are called *endpoints* of the sect AB.

II 5. Let A, B, C be three points not co-straight and a a straight in the plane ABC striking none of the points A, B, C: if then the strsight a goes through a point within the sect AB, it must always go either through a point of the sect BC or through a point of the sect AC."

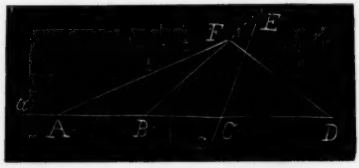
A year ago, while preparing my Report, I discussed with my pupil, R. L. Moore, the interpretation of the term "angeordnet" in II 4, and whether II 4 might not be demonstrated from the other assumptions.

Finally I wrote to Hilbert asking if he recognized the desirability of any change, as I wished to use his axioms as basis for a text-book on geometry.

His answer bears date April 2, 1902, and was received April 14, 1902. It begins: "Ueber Ihre Idea aus meinen Grundlagen eine Schul-Geometrie zu machen, bin ich sehr erfrunt." Finally, at the very end of the letter, in a postscript, Hilbert says; "Instead of II 4 I believe it suffices simply to say: If B lies between A and C and C between A and D, then lies also B between A and D; and then to prove my old II 4 as theorem."

I read this to Mr. Moore, and suggested his filling in the proof. Early next morning he announced to me that he had demonstrated Hilbert's new axiom, thus totally eliminating II 4, and reducing the Betweenness Assumptions from five to four. Mr. Moore has no intimation that any one has ever tried to prove these theorems. He started simply from the fact that two weeks ago Hilbert thought an assumption necessary of which now the demonstration, as I have written it out from his oral communication, seems of most unexpected simplicity and elegance.

THEOREM I. If B is between A and C, and C is between A and D, then C is between B and D.



PROOF. Let A, B, C, D be on a. Through C take a straight c other than a. On c take a point E other than C. On the straight BE, between B and E take F.

Thus between B and F is no point of c. Now between A and F there can be no point of c, else c would (by II 5) have a point between A and B, since by the construction of F, c cannot have point between B and F. Thus C would be between A and B, contrary to our hypothesis that B is between A and C.

Thus, since c cannot have a point between A and F, it must (by II 5) have a point between F and D. So we have the three non-co-straight points F, B, D, and c with a point between F and D, and, by construction, none between F and B. Therefore it must (by II 5) have a point between F and D. So C is between B and D.

THEOREM II. [Hilbert's new axiom, II 4']. If B is between A and C, and C is between A and D, then B is between A and D.

PROOF. Let A, B, C, D be on a. Through B take a straight b other than a. On b take a point E other than B. On the straight CE, between C and E take F. Thus between C and F is no point of b. Then since by hypothesis B is between A and C, therefore b must (by II 5) have a point between A and F. Thus we have the three non-co-straight points A, F, D, and b with a point between A and F. Therefore b must (by II 5) have a point between A and D or between F and D. But it cannot have a point between F and D, for then it must (by II 5) have a point either between F and C, contrary to our construction, or else between C and D, contrary to Theorem I, by which C is between B and D. Therefore it has a point between A and D. So B is between A and D.

THEOREM III. [Hilbert's old axiom II 4]. Any four points of a straight can always be so lettered $ABCD$, that B is between A and C and also between A and D, and furthermore C is between A and D and also between B and D.

PROOF. We may (by II 3) letter three of our points B, C, D, with C be-

tween B and D. Now as regards B and D, and our fourth point A, either A is between B and D, or B is between A and D, or D is between A and B.

If B is between A and D, we have fulfilled the hypothesis of Theorems I and II. If D is between A and B, then interchanging the lettering for B and D, that is, calling B, D, and D, B, we have again fulfilled the hypothesis of Theorems I and II.

There only remains to consider the case where A is between B and D. If now C is between D and A we have fulfilled the hypothesis of Theorems I and II, by calling D, A, and C, B, and A, C, and B, D. If however A were between C and D we would have fulfilled the hypothesis of Theorems I and II by writing for A, B, for D, A, and for B, D.

We have only left one case to consider, that where D is between A and C. This case is impossible. Suppose $ABCD$ on a. Through C take a straight c other than a. On c take a point E other than C. On the straight DE between D and E take F. Thus between D and F is no point of c.

Then since by hypothesis C is between B and D, therefore c must (by II 5) have a point between B and F. Therefore we have the three non-co-straight points B, F, A, and c with a point between B and F. Therefore c has (by II 5) a point between B and A or a point between F and A.

But it cannot have a point between F and A, else it would (by II 5) have a point between F and D, contrary to our construction, or else between D and and A, giving C between D and A, contrary to our hypothesis D between A and C. So C would be between B and A, and D between A and C, and therefore (by Theorem II) D between A and B, contrary to our hypotheris A between B and D.

Thus there is always such a lettering that B is between A and C, and C between A and D, whence (by Theorem I) C is between B and D, and (by Theorem II) B is between A and D.

Austin, Texas, April 17, 1902.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

155. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A bought a horse, which he sold to B at a *loss* of $m=6\%$; B sold the horse to C at a *loss* of $n=6\%$; and C sold the horse to D at a *gain* of $p=12\frac{1}{2}\%$. How much did A *lose*, if C gained $\$G=\$26.79$?

Solution by J. R. HITT, Coronal Institute, San Marcos, Tex.

Let 100% = what horse cost A. Then $\frac{(100-6)(100-5)\times 12\frac{1}{2}}{(100)^2}$

$=94\times(.95)(.12\frac{1}{2})=.111625$ of what horse cost A = C's gain = \$26.79.

Hence A's loss $=\frac{\$26.79\times.06}{.111625}=\$14.40$.

Also solved by *G. B. M. ZERR*.

ALGEBRA.

136. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve $a^{x^2}b^{y^2}=c$....(1), and $c^{x+y}=ab$....(2).

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

Let $\log a=m$, $\log b=n$, $\log c=p$. Then $mx^2+ny^2=p$, and $px+py=m+n$. From which we easily get

$$x=\frac{n}{p}\pm\frac{1}{p}\sqrt{\frac{p^3-mn(m+n)}{m+n}},\quad y=\frac{m}{p}\mp\frac{1}{p}\sqrt{\frac{p^3-mn(m+n)}{m+n}}.$$

Solved in a similar manner by *H. C. WHITAKER*, and *L. C. WALKER*.

137. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Solve, if possible, $a^x+b^x=c$.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; LON C. WALKER, A. M., Petaluma High School, Petaluma, Cal.; and F. P. MATZ, Sc. D., Ph. D., Defiance College, Defiance, Ohio.

Let $\log a=m$, $\log b=n$. Then

$$a^x=1+mx+\frac{m^2x^2}{2!}+\frac{m^3x^3}{3!}+\frac{m^4x^4}{4!}+\ldots.$$

$$b^x=1+nx+\frac{n^2x^2}{2!}+\frac{n^3x^3}{3!}+\frac{n^4x^4}{4!}+\ldots.$$

Adding, $c-2=(m+n)x+\frac{m^2+n^2}{2!}x^2+\frac{m^3+n^3}{3!}x^3+\frac{m^4+n^4}{4!}x^4\ldots.$

By reversion of series,

$$x=\frac{c-2}{m+n}-\frac{(m^2+n^2)(c-2)^2}{(m+n)^3\,2!}+\frac{[3(m^2+n^2)^2-(m+n)(m^3+n^3)](c-2)^3}{(m+n)^5\,3!}$$

$$-\frac{[15(m^2+n^2)^2-10(m+n)(m^2+n^2)(m^3+n^3)+(m+n)^2(m^4+n^4)](c-2)^4}{(m+n)^7\,4!}+\ldots$$

Also solved by *WM. E. HEAL*, and *J. SCHEFFER*.

138. Proposed by HARRY S. VANDIVER, Bala, Pa.

Show that the number of solutions in positive integers for x, y, and z of $x^3+2y^3+4z^3-6xyz=1$ is infinite.

Solution by the PROPOSER.

Suppose, for the present, that there is one solution of the proposed equation. Let $x=a$, $y=b$, $z=c$, satisfy it. Then

$$a^3+2b^3+4c^3-6abc=1. \quad \text{Whence } [a^3+2b^3+4c^3-6abc]^n=1.$$

Then whatever values of x, y, and z we take consistent with $x^3+2y^3+4z^3-6xyz=1$, we will have

$$[a^3+2b^3+4c^3-6abc]^n=x^3+2y^3+4z^3-6xyz,$$

which is satisfied by the following assumptions:

$$\begin{aligned} x+y\sqrt[3]{2}+z\sqrt[3]{4}&=[a+b\sqrt[3]{2}+c\sqrt[3]{4}]^n \\ x+\omega y\sqrt[3]{2}+\omega^2 z\sqrt[3]{4}&=[a+\omega b\sqrt[3]{2}+\omega^2 c\sqrt[3]{4}]^n \quad (1) \\ x+\omega^2 y\sqrt[3]{2}+\omega z\sqrt[3]{4}&=[a+\omega^2 b\sqrt[3]{2}+\omega c\sqrt[3]{4}]^n \end{aligned}$$

(where $\omega^3=1$) as may be seen by multiplying together the equations, term for term. (1) gives on expanding

$$x+y\sqrt[3]{2}+z\sqrt[3]{4}=A+B\sqrt[3]{2}+C\sqrt[3]{4}$$

where A, B, and C are positive integers.

Hence, we must have $x=A$, $y=B$, $z=C$.

(2) and (3) give likewise

$$\begin{aligned} x+\omega y\sqrt[3]{2}+\omega^2 z\sqrt[3]{4}&=A+\omega B\sqrt[3]{2}+\omega^2 C\sqrt[3]{4} \\ x+\omega^2 y\sqrt[3]{2}+\omega z\sqrt[3]{4}&=A+\omega^2 B\sqrt[3]{2}+\omega C\sqrt[3]{4} \end{aligned}$$

In each of these we will have as in (1) $x=A$, $y=B$, $z=C$.

Therefore, x, y, and z have an infinite number of integral values depending on the value of n; provided, as we assumed at the start, that they have one set of values.

This one set of values is $x=1$, $y=1$, $z=1$; for $1^3+2\times3^3+4\times1^3-6=1$.

Let us find, for instance, the values of x, y, and z for $n=2$, having given that $a=1$, $b=1$, $c=1$, we have

$$x+y\sqrt[3]{2}+z\sqrt[3]{4}=[1+\sqrt[3]{2}+\sqrt[3]{4}]^2, \text{ or } x=5,\ y=4,\ z=3,$$

which satisfy. Putting $n=3$, we find another set, and so on.

The method used in the solution of this problem is a particular case of a general principle applicable to the solution of many Diophantine questions.

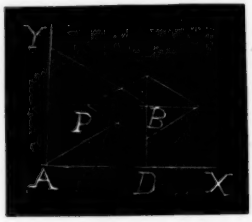
GEOMETRY.

167. Proposed by JOHN M. QUINN, Professor of Mathematics, High School, Warren, Pa.

If at the vertex of an isosceles triangle one of whose basal vertices is pivoted and the other free to move in a straight line, a rhombus be pivoted with sides parallel to the sides of the triangle, the locus of every point on the rhombus except the one which is its intersection with the fixed side of the triangle is an ellipse.

Solution by the PROPOSER.

Notation. Let X and Y be rectangular axes; θ the angle BAX; m equal segments of the sides of the triangle and the rhombus; $AB=a$, $BD=y$, and $AD=x$, and YBA an isosceles triangle with one basal vertex pivoted at A. The other basal vertex is free to move along the line AY.



To prove that the locus of any point as P is an ellipse.

PROOF. $y/a=\sin\theta$. $\therefore\ y^2/a^2=\sin^2\theta$.

$x=a\cos\theta-2m\cos\theta=\cos\theta(a-2m)$.

$$\frac{x}{a-2m}=\cos\theta,\quad \frac{x^2}{(a-2m)^2}=\cos^2\theta.$$

$$\therefore\ \frac{y^2}{a^2}+\frac{x^2}{(a-2m)^2}=\sin^2\theta+\cos^2\theta=1.$$ $\therefore$ the locus of p is an ellipse.

Similarly for any other point *mutatis mutandis*. Q. E. D.

168. Proposed by MISS GUBELMAN, Student Southern Illinois State University, Carbondale, Ill.

To draw a perpendicular to one side of a triangle dividing it into two equivalent parts.

Solution by the PROPOSER.

1. Let ABC be the triangle. Draw the median AD and the perpendicular AE. Construct BX such that $BX^2=BD.BE$. Draw the perpendicular XY.

$\therefore\ \triangle BXY:\triangle BAE=BX^2:BE^2$.

Similar triangles$=BD\times BE=BE^2=BD:BE$.

Also $\triangle BAD:\triangle BAE=BD:BE$, having equal altitudes.

$\therefore\ \triangle BXY:\triangle BAE=\triangle BAD:\triangle BAE$.

$\therefore\ \triangle BXY=\triangle BAD$. But $\triangle BAD=\frac{1}{2}\triangle ABC$. Median.

$\therefore\ \triangle BXY=\frac{1}{2}\triangle ABC$. $\therefore\ XY$ is the required perpendicular.

Also solved by *G. B. M. ZERR, DANIEL B. NORTHRUP, L. C. WALKER, J. SCHEFFER, H. C. WHITAKER, H. B. PENHOLLOW*, and *P. W. WEBBER*.

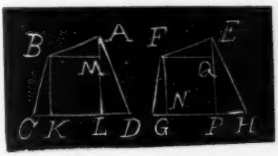
169. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Theorem. Two quadrilaterals having three sides of the one equal to the thre corresponding sides of the other, each to each, and the two corresponding angles adjacent to the unknown sides equal, each to each, are equal figures. [From *Olney's Geometry*, Section VIII, Proposition XIV].

1. Required proof. 2, Is this proposition found in any other text-book of Geometry?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $ABCD$, $EFGH$ be the two quadrilaterals; $BC=FG, AB=EF, AD=EH$, $\angle BCD=\angle FGH$, $\angle ADC=\angle EHG$. Draw BK, AL perpendicular to CD; FN, EP perpendicular to GH; BM perpendicular to AL; FQ perpendicular to EP.



Right triangles BCK and FGN are equal, also right triangles ALD and EPH, having hypotenuse and acute angle of one equal to hypotenuse and acute angle of other.

$\therefore$ $BK=FN$, $AL=EP$, also $AL-BK=AM=EP-FN=EQ$.

$\therefore$ right triangles ABM=right triangle FEQ; since $AB=FE$ and $AM=EQ$,

$\therefore$ $BM=FQ$. $\therefore$ $BM=KL=FQ=NP$. $\therefore$ $BKLM=FNPQ$.

$\therefore$ $BCK+ADL+ABM+BKLM=FGN+EPH+FEQ+FNPQ$.

$\therefore$ $ABCD=EFGH$.

Also solved by J. SCHEFFER.

170. Proposed by CHARLES C. CROSS, Whaleyville, Va.

If p, q, r are the distances of the orthocenter from the sides, prove that

$$4\left[\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right]=\left[\frac{a}{p}+\frac{b}{q}-\frac{c}{r}\right]\left[\frac{a}{p}-\frac{b}{q}+\frac{c}{r}\right]\left[-\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right].$$

Solution by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

From well known theorems we have

$$\begin{array}{lll} a=2R\sin A & & p=2R\cos B\cos C \\ b=2R\sin B & \text{and} & q=2R\cos C\cos A \\ c=2R\sin C & & r=2R\cos A\cos B \end{array}$$

Whence $a/p=\tan B+\tan C$, $b/q=\tan C\tan A$, $c/r=\tan A+\tan B$.

Therefore, $a/p+b/q+c/r=2(\tan A+\tan B+\tan C)$.

$$-a/p+b/q+c/r=2\tan A$$
$$a/p-b/q+c/r=2\tan B$$
$$a/p+b/q-c/r=2\tan C.$$

Since $\tan A+\tan B+\tan C=\tan A\tan B\tan C$, the theorem is proved.

Also demonstrated by L. C. WALKER, J. SCHEFFER, H. C. WHITAKER, G. B. M. ZERR, and the PROPOSER.

171. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the nearest distance of the parabola $y^2=16x$ and the ellipse $16x^2+9y^2-160-144y+832=0$.

Solution by the PROPOSER.

Let the equation of the parabola be $y^2=4px$, and that of the ellipse

$$\frac{(y-\beta)^2}{b^2}+\frac{(x-\alpha)^2}{a^2}=1.$$

A normal to the ellipse expressed by the tangent of the angle it makes with the axis of x has the equation

$$y-\beta=m(x-\alpha)-\frac{(a^2-b^2)m}{\sqrt{(b^2m^2+a^2)}},$$

and in the case of the parabola $y=mx-2pm-pm^3$. Since the shortest distance is measured off on a common normal, the two equations should be identical, and therefore

$$\beta-m\alpha-\frac{(a^2-b^2)m}{\sqrt{(b^2m^2+a^2)}}=-2pm-pm^3.$$

From this equation m is to be found. It leads to an equation of the 8th degree, which for numerical values presents no difficulty.

Substituting now the equation of the normal in both the equations of the parabola and ellipse, we find the co-ordinates of the points of intersection. Denoting these by x', y', and x'', y'', we find for the shortest distance the expression $\sqrt{[(x'-x'')^2+(y'-y'')]^2}$.

The numerical equations given having a common value, x lying between 6 and 7, and intersecting, therefore present no suitable example.

Also solved by *G. B. M. ZERR.*

CALCULUS.

129. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Among all quadrilaterals inscribed in an ellipse, to determine that which contains the greatest area.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Walton furnishes a solution of this interesting problem, which for its elegance and simplicity I reproduce here.

Let the equation of the ellipse be $x^2/a^2+y^2/b^2=1$, and let the angular points of the quadrilateral be (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4). Then, u denoting the area of the quadrilateral,

$$2u=x_2y_1-x_1y_2+x_3y_2-x_2y_3+x_4y_3-x_3y_4+x_1y_4-x_4y_1.$$

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1,\quad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1,\quad \frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}=1,\quad \frac{x_4^2}{a^2}+\frac{y_4^2}{b^2}=1.$$

In order that u may be a maximum, we have, differentiating the last four equations, putting $du=0$, and using the method of indeterminate multipliers,

$$\frac{\lambda_1 x_1}{a^2}+y_4-y_2=0, \quad \frac{\lambda_1 y_1}{b^2}-x_4+x_2=0.$$

. . . . . .

Hence we have $\frac{x_1}{a^2}(x_4-x_2)=-\frac{y_1}{b^2}(y_4-y_2)$....(1),

$$\frac{x_2}{a^2}(x_1-x_3)=-\frac{y_2}{b^2}(y_1-y_3)....(2), \quad \frac{x_3}{a^2}(x_2-x_4)=-\frac{y_3}{b^2}(y_1-y_4)....(3),$$

$$\frac{x_4}{a^2}(x_3-x_1)=-\frac{y_4}{b^2}(y_3-y_1)....(4).$$

From (1) and (3) we have $\frac{y_3}{x_3}=\frac{y_1}{x_1}$....(5), and from (2) and (4), $\frac{y_4}{x_4}=\frac{y_2}{x_2}$....(6). Also from (1) and (2), $\frac{1}{a^2}(x_1x_4-x_2x_3)=-\frac{1}{b^2}(y_1y_4-y_2y_3)$.

From the last three equations we see that

$$\frac{1}{a^2}(x_1x_4-x_2x_3)=-\frac{1}{b^2}y_1y_2\left[\frac{x_4}{x_2}-\frac{x_3}{x_1}\right], \text{ and therefore } \frac{y_1y_2}{x_1x_2}=-\frac{b^2}{a^2}..(7).$$

The equations (5) and (6) show that the diagonals of the quadrilaterals are diameters of the ellipse, and (7) shows that they are conjugate diameters.

Also solved by *G. B. M. ZERR, A. H. HOLMES*, and *L. C. WALKER*.

130. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Solve the differential equation $x^r(\frac{dy}{dx}+y\log x)-a=0$.

I. Solution by W. E. HEAL, Marion, Ind.

Writing the equation $\frac{dy}{dx}+y\log x=a/x^r$, we have a linear equation of the standard form, and by the well known formula, $y=-Ca/x^r$.

II. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Dividing by x^r and transposing we get $dy/dx+\log x.y=ax^{-r}$, the regular form of the *linear* equation.

Using Bernoulli's method, we proceed as follows:

Let $y=uz$; then $dy/dx=u\frac{dz}{dx}+z\frac{du}{dx}$, and the given equation becomes

$$u\frac{dz}{dx}+z(\frac{du}{dx}+u\log x)=ax^{-r}.$$

Now, if the function u be assumed of such value as to make the factor z in this equation vanish, we shall have the two equations,

$$\frac{du}{dx}+u\log x=0, \text{ and } u\frac{dz}{dx}=ax^{-x}.$$

From the former, we get by integration, $u=e^x x^{-x}$, and the value of u reduces the latter to $e^x\frac{dz}{dx}=a$ or $dz=ae^{-x}dx$. Whence $z=-ae^{-x}+C$.

Therefore $y=uz=e^x x^{-x}(-ae^{-x}+C)=x^{-x}(Ce^x-a)$, which is the required solution.

Also solved by *L. C. WALKER, G. B. M. ZERR, F. P. MATZ*, and *J. SCHEFFER*.

131. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Integrate $2/r$, with regard to $d[\sqrt{(1-x^2)}]$.

Solution by the PROPOSER.

According to the conditions of the problem, we may write
$d[F(x)]/d[\sqrt{(1-x^2)}]=2/x$....(1).

$$\therefore F(x)=2\int\frac{d[\sqrt{(1-x^2)}]}{x}=-2\int\frac{dx}{\sqrt{(1-x^2)}}=\sec^{-1}\left[\frac{1}{2x^2-1}\right]$$

$=\cos^{-1}(2x^2-1)$....(2), which is the integral required.

Also solved with various results by *H. C. WHITAKER, J. SCHEFFER*, and *G. B. M. ZERR*.

132. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

What expression derived from the *polar* equation of a curve is equivalent to the expression for dy/dx derived from the *Cartesian* equation of the same curve? Prove work with $\rho=2r\cos\theta$.

Solution by LON C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Let $x=\rho\cos\theta$, $y=\rho\sin\theta$, then

$$\frac{dy}{dx}=\frac{\sin\theta(d\rho/d\theta)+\rho\cos\theta}{\cos\theta(d\rho/d\theta)-\rho\sin\theta}....(1).$$

From $\rho=2r\cos\theta$, we get $d\rho/d\theta=-2r\sin\theta$. Substituting this value of $d\rho/d\theta$ in (1), and reducing, we have

$$\frac{dy}{dx}=\frac{r-x}{y}....(2).$$

Now substituting the value of x and y in $\rho=2r\cos\theta$, we get $x^2+y^2=2rx$; from which $dy/dx=(r-x)/y$, the same as in (2).

The angle the tangent makes with the radius vector is $\rho(d\theta/d\rho)$, which is often used in the same way as dy/dx; for instance, in trajectories, etc.

Also solved by *G. B. M. ZERR*, and *J. SCHEFFER*.

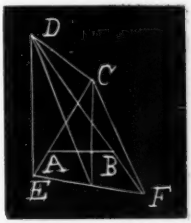
MECHANICS.

130. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Two particles are projected from A and B on the same level at α, β to horizon, and in vertical planes with which AB makes angles θ, φ. They meet and coalesce into a single particle. Find the height of the latus rectum of the subsequent path above the level of A and B.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let D be the point of meeting of the two particles; mass of each, unity; C, the projection of D on the plane of AB; DE, DF, the tangents to the two paths, respectively; G, the mid-point of EF. Then DG is the direction of the resultant of DE and DF. Let $AB=a$, $\angle CAB=\theta$, $\angle CBA=\varphi$.



Then $AC=a\sin\theta/\sin(\theta+\varphi)$, $BC=a\sin\varphi/\sin(\theta+\varphi)$.

The equation to the path through AD is $m=n\tan\alpha - gn^2/2v^2\cos^2\alpha$, when $n=a\sin\theta/\sin(\theta+\varphi)$.

$$m=\frac{a\sin\theta\tan\alpha}{\sin(\theta+\varphi)}-\frac{ga^2\sin^2\theta}{2v^2\sin^2(\theta+\varphi)\cos^2\alpha}=h\ldots.(1).$$

The equation to the path through BD is $p=q\tan\beta-gq^2/2v_1^2\cos^2\beta$, when $q=a\sin\varphi/\sin(\theta+\varphi)$;

$$p=\frac{a\sin\varphi\tan\beta}{\sin(\theta+\varphi)}-\frac{ga^2\sin^2\varphi}{2v_1^2\sin^2(\theta+\varphi)\cos^2\beta}=h\ldots.(2).$$

From (1) and (2),

$$v_1^2=\frac{agv^2\sin^2\varphi\cos^2\alpha}{\cos^2\beta[ga\sin^2\varphi+2v^2\cos^2\alpha\sin(\theta+\varphi)(\sin\varphi\tan\beta-\sin\theta\tan\alpha)]};$$

$$\tan DEC=dm/dn=\tan\alpha-gn/v^2\cos^2\alpha=\tan A,$$

$$\therefore\tan A=\tan\alpha-\frac{ag\sin\theta}{v^2\sin(\theta+\varphi)\cos^2\alpha};\ \tan DCF=\frac{dp}{dq}=\frac{\tan\beta-gq}{v_1^2\cos^2\beta}=\tan B,$$

$$\therefore\tan B=\tan\beta-\frac{ag\sin\varphi}{v_1^2\sin(\theta+\varphi)\cos^2\beta}.$$

Let velocity at D for particle from $A=u$, for particle from $B=u_1$; then $u^2=v^2-2gh$, $u_1^2=v_1^2-2gh$. Now $CE=h\cot A$, $CF=h\cot B$.

$\therefore EF=h\sqrt{[\cot^2 A+\cot^2 B+2\cot A\cot B\cos(\theta+\varphi)]}=d$.

$CG=\frac{1}{2}\sqrt{[2CE^2+2CF^2-EF^2]}=\frac{1}{2}h\sqrt{[\cot^2 A+\cot^2 B-2\cot A\cot B\cos(\theta+\varphi)]}=l$.

$\therefore \tan DGC=h/l=\tan C$, where C=angle of projection of coaleseed particles at D. Also $DE=h\operatorname{cosec} A$, $DF=h\operatorname{cosec} B$.

$$\therefore \cos EDF=\frac{h^2\operatorname{cosec}^2 A+h^2\operatorname{cosec}^2 B-d^2}{2h^2\operatorname{cosec} A\operatorname{cosec} B}=\cos D.$$

$\therefore \cos D=\sin A\sin B-\cos A\cos B\cos(\theta+\varphi)$.

Let w=the velocity of the two particles after they coalesce.

Then $w=\sqrt{[u^2+u_1^2+2uu_1\cos D]}$.

$\therefore$ Their path is $y=x\tan C-gx^2/2w^2\cos^2 C$.

Ordinate of vertex$=w^2\sin^2 C/2g$.

Latus rectum$=-2w^2\cos^2 C/g$.

Height of latus rectum above $D=\frac{w^2}{2g}(\sin^2 C-\cos^2 C)=\frac{w^2}{2g}(1-2\cos^2 C)=k$

Height of latus rectum above plane of $AB=h+k$.

131. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College. Defiance. Ohio.

If the distributed weight on the foundations of a building is W lb./(feet)^2, the foundations must be sunk $D=(W/w)\tan^4(\frac{1}{4}\pi-\frac{1}{2}\phi)$ feet deep in earth of density $w$lb./(feet)^3 and angle of repose ϕ.

No solution of this problem has been received.

132. Proposed by T. U. TAYLOR, C. E., Department of Engineering, University of Texas. Austin, Tex.

1. A parabola, whose axis is vertical, is described on the vertical face of a reservoir wall. If the vertex O of the parabola is at the bottom of the wall, and the parabola intersects the surface in the points A, B, find the depth of the center of pressure of the water on the parabolic area ABO.

2. In the same problem find the center of pressure on the area included between the horizontal line through O, a vertical through B, and the curve OB.

Solution by P. H. PHILBRICK, C. E., Lake Charles, La.

The general formula for the center of pressure, is

$$\bar{x}=\frac{\int\int hx\,dxdy}{\int\int hdxdy}$$

in which h is the depth of any point below the surface.

Let O be the origin, its depth being a, and the ordinate on the surface $=b$.

Let $y^2=m^2x$ be the equation of the curve. In this case we have,

$$\bar{x}=\frac{\int_0^a\int_0^{mx^{\frac{1}{2}}}(a-x)xdxdy}{\int_0^a\int_0^{mx^{\frac{1}{2}}}(a-x)dxdy}=\frac{\int_0^a(a-x)x^{\frac{3}{2}}\,dx}{\int_0^a(a-x)x^{\frac{1}{2}}\,dx}=\left[\frac{\frac{2}{5}ax^{\frac{5}{2}}-\frac{2}{7}x^{\frac{7}{2}}}{\frac{2}{3}ax^{\frac{3}{2}}-\frac{2}{5}ax^{\frac{3}{2}}}\right]_0^a=\frac{3}{7}a.$$

Therefore, the center of pressure is $\frac{4}{7}a$ below the surface.

(2). To find the center of pressure on the outer figure. We have

$$\bar{x}=\frac{\int_0^a\int_0^{b-mx^{\frac{1}{2}}}(a-x)xdxdy}{\int_0^a\int_0^{b-mx^{\frac{1}{2}}}(a-x)dxdy}=\frac{\int_0^a(a-x)(b-mx^{\frac{1}{2}})xdx}{\int_0^a(a-x)(b-mx^{\frac{1}{2}})dx}$$

$$=\left[\frac{\frac{1}{2}abx^2-\frac{2}{5}amx^{\frac{5}{2}}-\frac{1}{3}bx^3+\frac{2}{7}mx^{\frac{7}{2}}}{abx-\frac{2}{3}amx^{\frac{3}{2}}-\frac{1}{2}bx^2+\frac{2}{5}mx^{\frac{5}{2}}}\right]_0^a.$$

But the curve gives $b^2=m^2a$ or $m=b/a^{\frac{1}{2}}$. Substituting this value of m, and reducing, we have, $\bar{x}=\frac{11}{49}a$. Hence the depth of the center of pressure$=\frac{38}{49}a$.

Also solved by *G. B. M. ZERR*, and *T. T. DAVIS*.

DIOPHANTINE ANALYSIS.

89. Proposed by J. H. DRUMMOND, LL. D., Portland, Me.

Show that in $2x^2+2y^2-z^2=\square$....(1),

$2x^2+2z^2-y^2=\square$....(2),

$2y^2+2z^2-x^2=\square$....(3),

any two numbers and their sum and difference will satisfy the conditions.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $2x^2+2y^2-z^2=a^2$, $2x^2+2z^2-y^2=b^2$, $2y^2+2z^2-x^2=c^2$.

Adding, $2x^2+2y^2+2z^2=\frac{2}{3}(a^2+b^2+c^2)$.

$\therefore z=\pm\frac{1}{3}\sqrt{(2b^2+2c^2-a^2)}$,

$y=\pm\frac{1}{3}\sqrt{(2a^2+2c^2-b^2)}$,

$x=\pm\frac{1}{3}\sqrt{(2a^2+2b^2-c^2)}$.

Let $b=(n+1)a$, $c=na$.

$$\therefore z=\pm\frac{a}{3}(2n+1),\ y=\pm\frac{a}{3}(n-1),\ x=\pm\frac{a}{3}(n+2).$$

Solutions of problem 87 were received from *H.S.VANDIVER*, *G. B. M. ZERR*, *J. H. DRUMMOND*, and *H. C.WHITAKER*; of problem 88. from *G. B. M. ZERR*, *J. H. DRUMMOND*, *J. SCHEFFER*, and *H. C. WHITAKER*.

90. Proposed by HARRY S. VANDIVER, Bala, Pa.

Prove that it is always possible to find an infinite number of positive integral values of x, y, and z, such that the relation $z^2=x^2+bxy+cy^2$ is satisfied, b and c being any integers whatever.

Solution by the PROPOSER.

If $x^2+bxy+cy^2=z^2$....(1), then solving for x,

$$x=\frac{-b\pm\sqrt{[b^2y^2-4(cy^2-z^2)]}}{2}.$$

If x is to be a positive integer it follows that $b^2y^2-4cy^2+4z^2=$a square. So that the solution of the original problem is reduced to a solution of

$$X^2+AY^2=Z^2....(2).$$

It may be shown directly that this relation possesses an infinite number of integral solutions for X, Y, and Z, $-A$ being any integer whatever.

$$(m^2+An^2)^k=X^2+AY^2$$
$$\text{for } (m+n\sqrt{-A})^k=X+Y\sqrt{-A}$$
$$\text{and } (m-n\sqrt{-A})^k=X-Y\sqrt{-A}$$

whence. by multiplication, $m^2+An^2=X^2+AY^2$.

Hence we may assume in (2) that $Z=m^2+An^2$; m and n being any numbers. Then

$$(m^2+An^2)^2=X^2+AY^2$$
$$\text{and } (m+n\sqrt{-A})^2=X+Y\sqrt{-A}.$$

Expanding, we have $m^2-n^2A+2mn\sqrt{-A}=X+Y\sqrt{-A}$. Whence

$$X=m^2-n^2A$$
$$Y=2mn$$
$$\text{and by assumption, } Z=m^2+An^2.$$

Hence, (2) has an infinite number of solutions, since a solution is found for every value given to m and n. Also, for every solution of (2) there is a solution of (1), hence (1) has an infinite number of solutions, which was to be proved.

Proceeding in like manner, it can be shown that the relation

$$x^2+bxy+cy^2=z^n....(3)\ (n, \text{ an integer})$$

also possesses an indefinite number of solutions, but the method which has been used does not enable us to find *all* the solutions of either (1) or (3).

91. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

There are two unequal square numbers the sum of whose sum, difference, product, and quotient, is a square. Find the two numbers.

Solution by J. H. DRUMMOND, LL. D., Portland, Me.

Any two unequal squares answer the conditions of the question; for, let a^2 and b^2 be the numbers, then $a^2+b^2+a^2-b^2+a^2b^2+(a^2/b^2)$ must be a square. Reducing, and dropping the factor a^2, we have $2+b^2+(1/b^2)$, a square. This is readily put under the form $\frac{b^4+2b^2+1}{b^2}$, which is the square of $\frac{b^2+1}{b^2}$.

Q. E. D.

Also solved in a similar manner by *G. B. M. ZERR*, *J. SCHEFFER*, and *H. S. VANDIVER*.

92. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

(*a*) Find the least three integral numbers such that the difference of every two of them shall be a square number; (*b*) Find the least three square numbers such that the difference of every two of them shall be a square number.

Solution by J. H. DRUMMOND, LL. D., Portland, Me.

Part 2. Let x^2, m^2x^2, and n^2x^2 be the three numbers; then m^2-1, n^2-1, and m^2-n^2 must all be squares.

Assume $m=\frac{p^2+1}{p^2-1}$ and $n=\frac{q^2+1}{q^2-1}$, and m^2-1 and n^2-1 will be squares and it remains to make $\left[\frac{p^2+1}{p^2-1}\right]^2-\left[\frac{q^2+1}{q^2-1}\right]=\square$, or reducing $(p^2q^2-1)(\frac{q^2}{p^2}-1)=\square$. This is done by making $pq=\frac{r^2+s^2}{2rs}$ and $\frac{q}{p}=\frac{t^2+u^2}{2tu}$.

But $pq\times q/p=q^2=\left[\frac{r^2+s^2}{2rs}\right]\left[\frac{t^2+u^2}{2tu}\right]$. Hence $rs(r^2+s^2)tu(t^2+u^2)$ must be a square. Take $r=f+g$, $s=f-g$, $t=h+k$, and $u=h-k$, and substituting and reducing, we have $(f^4-g^4)(h^4-k^4)=\square$. Assume $f^2=h^4-k^4+v^2$ and $g^2=h^4-k^4-v^2$ and $(f^4-g^4)=4v^2(h^4-k^4)$, and $(f^4-g^4)(h^4-k^4)$ becomes $4v^2(h^4-k^4)^2$, a square. Assume $h^2=a^2v^2$ and $k^4=b^2v^2$; then $h^2=av$ and $k^2=bv$. But $f^2-g^2=2v^2$. Assume $f+g=4v$ and $f-g=\frac{1}{2}v$, then $f=\frac{9}{4}v$ and $g=\frac{7}{4}v$. Then $r=4v$ and $s=\frac{1}{2}v$. Then $pq\left[=\frac{r^2+s^2}{2rs}\right]=\frac{65}{16}$, and $q/p=\frac{t^2+u^2}{2tu}$ $=\frac{2(h^2+k^2)}{2(h^2-k^2)}=\frac{a+b}{a-b}$; and $h^4-k^4+v^2=f^2=\frac{81v^2}{16}$, and $h^4-k^4=\frac{65v^2}{16}$. Therefore $a^2-b^2=\frac{h^4-k^4}{v^2}=\frac{65}{16}$. Now $pq\times q/p=q^2=\frac{65(a+b)}{16(a-b)}=\frac{65(a+b)^2}{16(a-b)^2}=(a+b)^2$ and $q=(a+b)$ and $p=(a-b)$. Hence $m=\frac{(a-b)^2+1}{(a-b)^2-1}$ and $n=\frac{(a+b)^2+1}{(a+b)^2-1}$, in which a and b may be any square numbers which make $a^2-b^2=\frac{65}{16}$.

It would seem that $a^2-b^2=\frac{65}{16}$ ought to lead readily to a general solution, but a and b were both so taken that they must be squares; b is readily found to

be $\frac{13-5c^2}{8c}$. Hence $26c-10c^3$ must be a square; it is evident that this is the case when $c=1$. Then $b=1$ and $a=\frac{3}{4}$. Substituting these in the values of m and n, and we have $m=\frac{4}{9}\frac{1}{5}$ and $n=\frac{1}{1}\frac{8}{5}\frac{5}{3}$. Taking $x=153$, we have $mx=697$, and $nx=185$, and the numbers are $(153)^2$, $(185)^2$, and $(697)^2$.

Also solved by *G. B. M. ZERR*, and *EDWARD D. GRABER*.

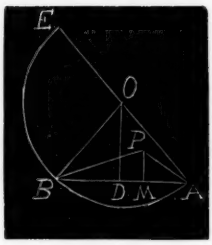
AVERAGE AND PROBABILITY.

111. Proposed by LON C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

If a radius be drawn at random in a given semi-circle, and a point taken at random in one of the sectors formed, show that the chance that a random line drawn through the point will cut the arc of the sector is $1-\frac{1}{\pi^2}\log 2$.

Solution by the PROPOSER.

Let ABE be the given semicircle, OB the random radius, P the random point, OD and PM perpendicular to AB.



Put $DM=x$, $PM=y$, $OA=1$, $\angle AOB=\theta$, $\angle APM=\phi$, $\angle BPM=\psi$. Then $AD=\sin\frac{1}{2}\theta$, $OD=\cos\frac{1}{2}\theta$, area of segment $ACB=\frac{1}{2}(\theta-\sin\theta)$, $\phi=\tan^{-1}\left(\frac{\sin\frac{1}{2}\theta-x}{y}\right)$, $\psi=\tan^{-1}\left(\frac{\sin\frac{1}{2}\theta+x}{y}\right)$

When P is in the segment ACB the random line will cut the arc whatever be its direction, and when P is in the triangle AOB the number of favorable directions of the random line will be $2(\phi+\psi)$. Hence we have

$$p=\frac{\int_0^{\pi}\pi(\theta-\sin\theta)d\theta+\int_0^{\pi}\int_0^{\cos\frac{1}{2}\theta}\int_{-\tan\frac{1}{2}\theta(\cos\frac{1}{2}\theta-y)}^{\tan\frac{1}{2}\theta(\cos\frac{1}{2}\theta-y)}2(\phi+\psi)dxdyd\theta}{\int_0^{\pi}\pi\theta d\theta}$$

$$=1-\frac{4}{\pi^2}+\frac{4}{\pi^3}\int_0^{\pi}\int_0^{\cos\frac{1}{2}\theta}\left[2\tan\tfrac{1}{2}\theta(2\cos\tfrac{1}{2}\theta-y)\tan^{-1}\tan\tfrac{1}{2}\theta\left(\frac{2\cos\frac{1}{2}\theta-y}{y}\right)\right.$$

$$\left.-y\theta\tan\tfrac{1}{2}\theta-y\log\left(\frac{y^2+\tan^2\frac{1}{2}\theta(2\cos\frac{1}{2}\theta-y)^2}{2y^2}\right)\right]dy\,d\theta$$

$$=1-\frac{4}{\pi^2}+\frac{4}{\pi^3}\int_0^{\pi}\left[\theta\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-2(\theta-\pi)\sin^3\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-2\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta\log^2\right.$$

$$\left.+\tfrac{1}{4}\log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)+\tfrac{1}{4}\cos\theta\log(1+\cos\theta)+\tfrac{1}{2}\cos2\theta\log(1-\cos\theta)\right]d\theta=1-\frac{1}{\pi^2}\log 2.$$

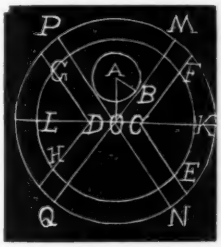
Solved with same result by *F. P. MATZ*. Professor Zerr gets as a result $1-(1/4\pi^2)(8\log 2+7)$.

112. Proposed by LON C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Two circles are drawn at random, both in magnitude and position, but so as to lie wholly upon the surface of a given circle. Show that the chance of their both resting on the same diameter of the given circle is $\frac{4}{\pi}(\log 2-5)$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let A be the center of one of the circles, $AO=x$, $AB=y$, $z=$radius of other circle. With center O and radius $OK=r-z$ describe a circle. Draw MQ, NP through O tangent to circle center A.



Also draw CE, DH, CF, DG parallel to NP, MQ, respectively, at a distance z from them. If center of the second circle lies on either $CEKFC$ or $GDHLG$, both circles will not lie on the same diameter.

Area $CEKFC$+area $GDHLG=$

$$2\left[(r-z)^2\left[\cos^{-1}(y/x)-\sin^{-1}\left(\frac{z}{r-z}\right)\right]-z\sqrt{(r^2-2rz)}+\frac{yz^2}{\sqrt{(x^2-y^2)}}\right]=u.$$

The limits of z are 0 and $r\sqrt{(x^2-y^2)}/[x+\sqrt{(x^2-y^2)}]=z'$; of y, 0 and r, and 0 and $(r-x)$; of x, 0 and $\frac{1}{2}r$, and $\frac{1}{2}r$ and r.

$$\therefore p=1-\frac{\left[\int_{\frac{1}{2}r}^{r}\int_{0}^{r-x}2\pi x\,dx\,dy+\int_{0}^{\frac{1}{2}r}\int_{0}^{x}2\pi x\,dx\,dy\right]\int_{0}^{z'}u\,dz}{\int_{0}^{r}\int_{0}^{r-x}\int_{0}^{r}\pi(r-z)^2.2\pi x\,dx\,dy\,dz}$$

$$=1-\frac{18}{\pi r^6}\left[\int_{\frac{1}{2}r}^{r}\int_{0}^{r-x}x\,dx\,dy+\int_{0}^{\frac{1}{2}r}\int_{0}^{x}x\,dx\,dy\right]\int_{0}^{z'}u\,dz$$

$$=1-\frac{4}{5\pi r^3}\left[\int_{\frac{1}{2}r}^{r}\int_{0}^{r-x}\{15\cos^{-1}(y/x)-10+\frac{5}{y^3}[3x^2+3x\sqrt{(x^2-y^2)}-y^2]\right.$$

$$\times[x-\sqrt{(x^2-y^2)}]^3\}x\,dx\,dy+\int_{0}^{\frac{1}{2}r}\int_{0}^{x}\{15\cos^{-1}(y/x)-10$$

$$\left.+\frac{5}{y^3}[3x^2+3x\sqrt{(x^2-y^2)}-y^2][x-\sqrt{(x^2-y^2)}]^3\}x\,dx\,dy\right].$$

Let $y=x\sin\theta$.

Then $\theta=0$ and $\sin^{-1}\left[\frac{r-x}{x}\right]=\theta'$.

$$\therefore p=1-\frac{4}{5\pi r^3}\left[\int_{\frac{1}{2}r}^{r}\int_0^{\theta}[15(\tfrac{1}{2}\pi-\theta)\cos\theta-10\cos\theta+10\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-\tfrac{5}{2}\tan\tfrac{1}{2}\theta\sec^2\tfrac{1}{2}\theta]\times\right.$$

$$\left.x^2\,dx\,d\theta+\int_0^{\frac{1}{2}r}\int_0^{\frac{1}{2}\pi}[15(\tfrac{1}{2}\pi-\theta)\cos\theta-10\cos\theta+10\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-\tfrac{5}{2}\tan\tfrac{1}{2}\theta\sec^2\tfrac{1}{2}\theta]x^2\,dx\,d\theta\right]$$

$$=1-\frac{4}{5\pi r^3}\left[\int_{\frac{1}{2}r}^{r}\left(15(r-x)\cos^{-1}(\frac{r-x}{x})-10r-20\sqrt{(2rx-r^2)}\right.\right.$$

$$\left.\left.+\tfrac{65}{2}x-\frac{5x^2}{x+\sqrt{(2rx-r^2)}}\right)dx+\tfrac{15}{2}\int_0^{\frac{1}{2}r}x^2dx\right]=\frac{4}{\pi}(8\log 2-5).$$

Also solved with same result by the *PROPOSER*.

MISCELLANEOUS.

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107. Proposed by **WILLIAM HOOVER**, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

The index of refraction of a medium varying inversely as the square root of the distance, prove that the path of a ray of light in the medium is a cycloid.

Solution by the PROPOSER.

Taking the axis of y in the given plane and that of x at right angles to the y axis, letting $\mu=k/\sqrt{x}$ be the index of refraction, and $p=dy/dx$, we have, by the usual theory, for the differential equation to the path

$$\frac{dp/dx}{1+p^2}=\frac{1}{\mu}\left[\frac{d\mu}{dy}-\frac{d\mu}{dx}\frac{dy}{dx}\right]....(1).\quad\text{We have}\quad\frac{d\mu}{dx}=-\frac{k}{2x^{\frac{3}{2}}}....(2),$$

and (1) becomes $\dfrac{dp/dx}{1+p^2}=\dfrac{p}{2x}$, or $\dfrac{dp}{p(1+p^2)}=\dfrac{dx}{2x}....(3).$

Integrating, $\log\dfrac{p}{\sqrt{(1+p^2)}}=\log\sqrt{x}+C....(4).$

Let $p=b$, when $x=a$; then $C=\log\dfrac{b}{a\sqrt{(1+b^2)}}$,

and (4) becomes $p=\dfrac{dy}{dx}=\dfrac{xdx}{\sqrt{[(a^2/b^2)(1+b^2)x-x^2]}}....(5)$, the differential equation to a cycloid.

Also solved by *G. B. M. ZERR*, and *L. C. WALKER*.

108. Proposed by **G. B. M. ZERR**, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

To divide the arc of a cardioid into eight equal parts.

Solution by L. C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Equation of the cordioid is $r=a(1+\cos\theta)$, and its complete perimeter is

$$S=2\int_0^{\pi}\sqrt{[a^2(1+\cos\theta)^2+a^2\sin^2\theta]}d\theta=4a\int_0^{\pi}\cos\tfrac{1}{2}\theta d\theta=8a,$$

the arc S being measured from the point where the curve crosses the initial line.

Let θ' be the point of the first division; then

$$2a\int_0^{\theta'}\cos\tfrac{1}{2}\theta d\theta=4a\sin\tfrac{1}{2}\theta'=a;$$

from which $\theta_{1,\ 7}=\pm 2\sin^{-1}\frac{1}{4}$. Similarly, $\theta_{2,\ 6}=\pm 2\sin^{-1}\frac{1}{2}$, $\theta_{3,\ 5}=\pm 2\sin^{-1}\frac{3}{4}$, $\theta_4=\pi$, and $\theta_8=0$.

Excellent solutions were received from *G. B. M. ZERR*, and *F. P. MATZ*.

109. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the latitude of the place where the sun's center remains above the horizon for a hundred successive days.

Solution by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, O.

I. The solar phenomenon in question will begin May 2 and end August 10. For May 2 at Washington mean noon, according to the Director of United States Nautical Almanac Office, the declination of the sun is $\delta=+15^\circ\ 18'\ 54.7''$. Representing the hour-angle of the sun by h, we have $\cos h=-\tan\lambda\tan\delta$....(1).

Since there is to be no setting of the sun, we may put $\cos h=-1$.

$\therefore\ \lambda=\tan^{-1}\left(\frac{1}{\tan\delta}\right)=\tan^{-1}(\cot\delta)=90^\circ-\delta,=74^\circ\ 41'\ 5.3''$, which is the terrestrial latitude of the place required. This solar phenomenon may be observed on Melville Island.

II. The celestial longitude of the sun for mean noon Washington can be determined in various ways to be about $\phi=41^\circ\ 34'\ 25''$; the obliquity of the ecliptic may be taken $\omega=23^\circ\ 27'\ 8''$; then $\lambda=90-\delta=90-\sin^{-1}(\sin\phi\times\sin\omega),=74^\circ\ 43'\ 16''$.

Also solved by *G. B. M. ZERR*, and *H. C. WHITAKER*.

110. Proposed by E. W. MORRELL, South Trowbridge, Vt.

If a and b be the sides of a triangle, A and B the angles opposite, then will $\log b-\log a=\cos 3A-\cos 2B+\frac{1}{2}(\cos 4A-\cos 4B)+\frac{1}{3}(\cos 6A-\cos 6B)+....$

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

$1+\cos 2A+\frac{1}{2}\cos 4A+\frac{1}{3}\cos 6A+....=-\frac{1}{2}\log(2-2\cos 2A)=-\log(2\sin A)$.

$1+\cos 2B+\frac{1}{2}\cos 4B+\frac{1}{3}\cos 6B+....=-\log(2\sin B)$.

See Trigonometry for the summation of these series.

$\therefore\ \cos 2A-\cos 2B+\frac{1}{2}(\cos 4A-\cos 4B)+....=\log(\sin B/\sin A)=\log(b/a)$.

Also solved by *F. P. MATZ*, and *J. SCHEFFER*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

158. Proposed by JAMES F. LAWRENCE. A. B., Professor of Mathematics, Rogers Academy, Rogers, Ark.

My agent sold pork at 5% commission; increasing the proceeds by \$20, I ordered the purchase of flour at 3% commission; after which flour rose 9%, my whole gain was \$40. What did he sell the pork for?

159. Proposed by F. P. MATZ. Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

The amount of tax assessed on the property of a city is T,=\$145850; and the treasurer was allowed a fee of $m\%$,=$\frac{3}{4}\%$, for collection. If $n\%$,=10%, of the tax was uncollectible, what were the net proceeds of the tax?

ALGEBRA.

153. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

If $x=\Sigma_0^\infty e^{-k[t+(2a\pi/h)]}\sin n\left(t+\frac{2a\pi}{h}\right)$, find value of x freed from Σ_0^∞.

154. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Deduce the Sylvestrian Reciprocant from $ax^3+3bx^2y^2+ay^3+d=0$.

GEOMETRY.

187. Proposed by R. TUCKER, M. A.

AD, BE, CF are the altitudes of the triangle ABC; k_1, k_1'; k_2, k_2'; k_3, k_3' are the S points of the triangles EAB, FCA; FBC, DAB; DCA, EBC, respectively; prove that $k_3'k_1=k_1'k_2=k_2'k_3=R\sin A\sin B\sin C$. ρ_1, ρ_1'; ρ_2, ρ_2'; ρ_3, ρ_3' are the Brocard radii of the above triangles, prove that (1) $\rho_1\rho_2\rho_3=\rho_1'\rho_2'\rho_3'$; (2) $(\rho_2'^2-\rho_3^2)/a^2+(\rho_3'^2-\rho_1^2)/b^2+(\rho_1'^2-\rho_2^2)/c^2=\frac{3}{4}$; (3) the sets of 4 Brocard-points for the above pairs of triangles are concyclic (on three circles); (4) the tangent from any one of the right angles of the above triangles to the Brocard circle of the triangle is a mean proportional between the tangents to the same circle from the remaining (two) angles.

188. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

$ABCD$ is a quadrilateral whose diagonal triangle is PQR, P on AD and R on AB. PQ meets AB in Z. If C moves along PB what will happen to Z?

CALCULUS.

152. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, Ohio.

Solve the linear differential equation

$$e^x\left[\frac{dy}{dx}-y\log x\right]-a[\log x+1]=0.$$

153. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the equation of the loxodromic curve on an oblate spheroid.

MECHANICS.

142. Proposed by GEORGE R. DEAN, B. Sc., Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

In infinite mass of liquid is bounded by the plane zx, on which are small corrugations given by $y=\phi(x)$. The velocity of the liquid at an infinite distance from the plane is parallel to x and equal to V. Prove that the velocity potential is $V_x+\frac{V}{\pi}\int_{-\infty}^{\infty}\frac{(x-\lambda)\phi(\lambda)d\lambda}{y^2+(x-\lambda)^2}$. [Bassett's *Hydrodynamics.*]

143. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Beads are fastened at equal intervals on a string placed over a smooth fixed pulley. If the original position of the string is one of symmetry, find the velocity at any moment, the pressure on the pulley, and the velocity with which the string leaves the pulley.

DIOPHANTINE ANALYSIS.

102. Proposed by F. L. SAWYER, Mitchel, Ontario, Canada.

Prove that the factors of the sum of the squares of two numbers prime to each other are themselves the sum of two squares.

103. Proposed by HARRY S. VANDIVER, Bala, Pa.

Find some solutions of $x^3+ay^3=z^2$ (for x, y, and z) and show that there is an infinite number of solutions corresponding to each integral value of a.

AVERAGE AND PROBABILITY.

127. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

What is the probable error of the volume of a rectangular parallelopiped whose edges measured by the repeated application of a unit of measure are found to be a, b, c, supposing that the probable error of a line so measured whose length is found to be l is $r\sqrt{l}$.

128. Proposed by G. B. M. ZERR, A. M., Ph. D., Profeseor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Two small circles are drawn on the surface of a sphere so as to intersect; find average area of the spherical triangle formed by joining the poles and one of the intersections of the small circles with arcs of great circles.

BOOKS AND PERIODICALS.

Four-Place Logarithmic Tables containing the Logarithms of Numbers and of the Trigonometric Functions. Arranged for use in the entrance examinations of the Sheffield Scientific School of Yale University. By Percy F. Smith. New York: Henry Holt & Co.

Elementary Calculus. A text-book for the use of students in General Science. By Percy F. Smith, Ph. D., Professor of Mathematics in the Sheffield Scientific School of Yale University. 8vo. Cloth, 89 pages. Price, 80 cents. New York and Chicago: The American Book Co.

The aim of this book is to satisfy the growing demand for a text-book on the Calculus which shall present in a course of from thirty-five to forty exercises the fundamental notions of the Calculus. The remarkable development of Physics and Chemistry, not to mention other important subjects, along mathematical lines, makes a working knowledge of the Calculus an absolute necessity. This book can easily be mastered in the time assigned by any one having a slight aptitude for mathematics.

Physics for High School Students. By Henry S. Carhart, LL. D., Professor of Physics in the University of Michigan, and Horatio N. Chute, M. S., Instructor of Physics in the Ann Arbor High School. 8vo. Cloth, vii+433 pages. Price, $1.25. Boston: Allyn & Bacon.

This volume, the authors say, was written with the same purpose that guided them in the preparation of their *Elements of Physics.* The Elements was a most excellent book, well adapted to the wants of the schools for which it was written. In bringing out this book, the rapid advances in Physics within the last five or six years made it necessary to rewrite the whole book, and thus make it an entirely new one. The authors have made great improvements, and those teachers who, like myself, found the Elements a most excellent work for academic purpcses, will find this one even better and more helpful.

Annals of Mathematics. Founded by Ormond Stone. Published under the auspices of Harvard University. Second Series, Vol. 3, No. 3.

The April number contains: Space of Constant Curvature, by Prof. F. S. Wood; Brilliant Points and Loci of Brilliant Points, by W. H. Roever; Problems in Infinite Series and Definite Integrals, by Prof. W. F. Osgood; Note on the Product of Linear Substitutions, by Prof. H. B. Newson.

The American Journal of Mathematics. Edited by Frank Morley with the co-operation of Simon Newcomb and other mathematicians. Published under the auspices of Johns Hopkins University. Price, $5.00.

Number 2, Vol. XXIV, contains Canonical Form of Linear Homogeneous Transformation in an Arbitrary Realm of Rationality, by Dr. L. E. Dickson; A New Theory of Collineation and their Lie Groups, by H. B. Newson; Infinitesimal Deformation of Surfaces, by L. P. Eisenhart.

ERRATA.

Page 66, line 7, from top of page, for "oreer" read *order.*

Page 67, line 15, from top of page, for "$\alpha-\beta+\delta-\varepsilon$" read $\alpha-\beta+\gamma-\varepsilon$.

Page 70, line 1, for "$x=22$" read $x=17$.

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MODELS OF THE WEIERSTRASS SIGMA FUNCTION AND THE ELLIPTIC INTEGRAL OF THE SECOND KIND.

By DR. VIRGIL SNYDER, Cornell University.

In 1886, Professor Dyck, of the Polytechnicum of Munich, requested the construction of models to represent the elliptic functions. In accordance with this request H. Burkhardt and M. W. Wildbrett constructed seven models, two of which are for the case in which the invariant J of the binary quartic is equal to zero.

In 1899, Professor Klein, while lecturing on the automorphic functions, requested that the series be extended by making corresponding models of the sigma function, and of the elliptic integral of the second kind.

By Professor Klein's invitation I undertook the task which consists of three steps. These may be named as follows:

(1) Analytic representation;
(2) Numerical calculation;
(3) Mechanical construction.

(1) ANALYTIC REPRESENTATION. The sigma function can most easily be expressed in terms of theta functions, which in turn are defined by trigonometric series with exponential functions for coefficients. For values of the variable within the parallelogram containing the origin two terms of the series are sufficient for numerical values correct to three places of decimals. For values of the